Introduction to Dynamical Systems

Solutions Problem Set 6

Exercise 1. Let (X, m) be a probability space and σ be underlying sigma-algebra of measurable sets. Introduce

$$\tilde{X} = \sigma / \sim,$$

where the equivalence relation $\sim$ is defined via $A \sim B$ if and only if $m(A \triangle B) = 0$.

(i) Show that the function

$$d\colon \begin{array}{ccc} \tilde{X} \times \tilde{X} & \longrightarrow & \mathbb{R}_+ \\ ([A],[B]) & \longmapsto & m(A \triangle B) \end{array}$$

defines a metric on $\tilde{X}$.

(ii) Show that $(\tilde{X}, d)$ is complete.

Solution. The fact that d is a distance on $\tilde{X}$ is immediate, and in order to show (ii), notice that if $(A_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then every subsequence converges to the same limit, provided one such limit exists. Hence we can consider a subsequence $(A_n)_{n\in\mathbb{N}}$ satisfying

$$d(A_n, A_{n+1}) < 2^{-n}.$$

Let $U_n = \bigcup_{m\geq n} A_m$ and $L_n = \bigcap_{m\geq n} A_m$. We have $L_n \subset A_n \subset U_n$ and

$$\begin{aligned} d(A_n, U_n) &= m\left(\left(\bigcup_{m\geq n} A_m\right) \setminus A_n\right) + m\left(A_n \setminus \bigcup_{m\geq n} A_m\right) \\ &\leq m\left(\bigcup_{m\geq n} A_{m+1} \setminus A_m\right) \leq \sum_{m\geq n} 2^{-m} = 2^{1-n}. \end{aligned}$$

Similarly,

$$\begin{aligned} d(A_n, L_n) &= m\left(\left(\bigcap_{m\geq n} A_m\right) \setminus A_n\right) + m\left(A_n \setminus \bigcap_{m\geq n} A_m\right) \\ &\leq m\left(\bigcup_{m\geq n} A_m \setminus A_{m+1}\right) \leq \sum_{m\geq n} 2^{-m} = 2^{1-n}. \end{aligned}$$

Now consider $A = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m$. Then $L_n \subset E \subset U_n$ and therefore

$$d(A_n, A) \leq d(U_n, L_n) \leq d(U_n, A_n) + d(L_n, A_n) \leq 2^{2-n} \underset{n\to\infty}{\longrightarrow} 0.$$

$\square$

Exercise 2. Let (X, m) be a probability space and $T\colon X \longrightarrow X$ a measure preserving map. Assume that for each pair of measurable sets $A, B \in \sigma$ there exists N such that

$$m(A \cap T^{-n}(B)) = m(A)m(B), \quad \forall n \geq N.$$

Show that $m(A) \in \{0, 1\}$ for each $A \in \sigma$. *Hint: use Baire category theorem and (1).*

Solution. Assume for contradiction that there is $B \in \sigma$ with $0 < m(B) < 1$. Define, for each $n \geq 1$, the set

$$\mathcal{B}_n = \left\{ A \in \sigma \colon m(A \cap T^{-n}(B)) \neq m(A)m(B) \right\}.$$

Then $\mathcal{B}_n$ is open in the topology in **Exercise 1** for all $n \geq 1$ and so is $\mathcal{J}_k = \bigcup_{n \geq k} \mathcal{B}_n$. Moreover, it is dense.

Indeed, given a set $A \notin \mathcal{J}_k$ with $m(A) > 0$, choose $n_1 > k$ such that $m(A \cap T^{-n_1}(B)) = m(A)m(B)$, and let $A_1 := A \cap T^{-n_1} B$. Inductively, if $A_{\ell-1} \subset A$ with $m(A_{\ell-1}) = m(A)m(B)^{\ell-1}$ has already been defined, use the hypothesis to find $n_\ell > k$ such that $A_\ell := A_{\ell-1} \cap T^{-n_\ell} B$ satisfies

$$m(A_\ell) = m(A_{\ell-1})m(B) = m(A)m(B)^\ell.$$

Given $\varepsilon > 0$, choose ℓ such that $m(A_\ell) = m(A)m(B)^\ell < \varepsilon$. Then $d(A, A \setminus A_\ell) < \varepsilon$, and

$$m((A \setminus A_\ell) \cap T^{-n_\ell} B) = m(A \cap T^{-n_\ell} B) - m(A_\ell) = m(A)m(B) - m(A_\ell) \neq m(A \setminus A_\ell)m(B).$$

Thus $A \setminus A_\ell \in \mathcal{J}_k$, and for the same reason, $A_\ell \in \mathcal{J}_k$. To approximate the equivalence class of the empty set, use the above construction with Ω in place of A, to obtain a measurable set $\Omega_\ell \in \mathcal{J}_k$ with $m(\Omega_\ell) = m(B)^\ell < \varepsilon$. Thus $d(\Omega_\ell, \emptyset) < \varepsilon$, and we have verified that $\mathcal{J}_k$ is dense in (M, d).

By the Baire category theorem, the intersection $\bigcap_{k \geq 1} \mathcal{J}_k$ is nonempty, and any set A in it contradicts the hypotheses. □

Exercise 3. Let (X, m) be a finite measure space and $T \colon X \longrightarrow X$ a measure preserving, bijective map with measurable inverse T^{-1}. Then show that for every $f \in L^1(X, m)$ we have

$$\lim_{N \to \infty} \sum_{j=0}^{N-1} f(T^j(x)) = \lim_{N \to \infty} \sum_{j=0}^{N-1} f(T^{-j}(x))$$

for almost every $x \in X$.

Solution. Recalling the proof of the mean ergodic theorem in Lecture5.pdf, notice that the subspace $V = \{U_T g = g\}$ generated by T is the same as $V' = \{U_{T^{-1}} g = g\}$ induced by T^{-1}. Hence the orthogonal projection P_T is equal to $P_{T^{-1}}$. We conclude that the averages in the statement converge to the same function in $L^2(X, m)$. To show that this is the case in $L^1(X, m)$ as well, one follows a similar procedure as that of Exercise 4 in Problem Set 4. □